More on the Somigliana waves

(C,\(\alpha\) and P,\(\alpha\))

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SUMMARY. — Previously Caloi has shown how a few corrections and rectifications of a theory on surface waves published by Carlo Somigliana in 1917, permit a quite new interpretation of the classic Rayleigh equation. In the light of this new interpretation, contrary to the Rayleigh theory, the real roots greater than 1 of the velocity equation have a correct physical meaning. More than a period of thirty years of useless attempts, finally, have marked a strict theoretical justification about conspicuous systems of typical surface waves, which start at the bottom of the Earth’s crust layers, when the angles of incidence of the longitudinal waves or transverse waves are effective. These are the so-called \(P_\alpha\) and \(C_\alpha\) waves. These waves are very interesting because they exhibit a new mechanism of elastic waves propagation, and they permit to study the fundamental characteristics of the crust stratifications. Owing to the considerable wavelengths, Somigliana waves present appreciable advantages as compared to ordinary longitudinal waves or transverse waves, both they are not subjected to phenomena of anomalous dispersion (see the case of the ordinary body waves of high frequency), as well as for they indicate the existence of discontinuities in the layers of transition-moves, which generally escape the investigations when the same are performed by ordinary longitudinal or transverse waves.

Besides the new very clear examples of these particular types of waves — which, moreover, often follow closely the faults in which the Somigliana waves arise; they waves any start also at the greatest distances (like the \(C_\alpha\) waves), and sometimes they form the most large phase of a seismogram.
1. It has already been proved (2) that the Rayleigh equation tolerates real roots greater than 1 only for values of \( \alpha \) between 0 and 0.26305. In relation to this last value of \( \alpha \), the real roots greater than 1 coincide at the value 3.5754. For values of \( \alpha \) between 0.26305 and 0.5, the Rayleigh equation has a single real root, namely the one smaller than 1 (and to which, notably, correspond the so-called Rayleigh waves). The other two roots become complex. The Fig. 1 shows the real roots greater than 1 corresponding to the field of variability of \( \alpha \).

Theoretically, this should occur when, according to the Poisson ratio, we have the given limits:

\[
0 < \alpha < 0.26305
\]

Basing on experience, the question may be raised whether each root greater than 1 actually corresponds to Somigliana waves. In fact, with regard to the Earth, the limit is much narrower. We know that \( \alpha \) only rarely drops to values less than 0.25 (3). Therefore, values up to 0.1 have been found for single rocks (4).
practically, the roots which correspond to conductive angles are the ones offered by the Rayleigh equation for the following range of variability of \( a \):

\[ 0.35 < a < 0.20305. \]

At this point, another question arises. Is the value greater than 1 for the real root sufficient reason to generate the surface waves of the Somigliana type?

The answer requires a consultation of the expression assumed on the surface layers by the displacements associated with the mentioned waves.
It has been shown (*) that by designating with \( u, v, w_1 \) and \( M_2 \), respectively, the horizontal and the vertical components of the longitudinal (index 1) and vertical (index 2) wave, which when paired are responsible for the generation of Somigliana waves, we may write:

\[
\begin{align*}
    s_1 &= a_1 \left[ x \gamma + y \eta - \mu \right], \\
    s_2 &= a_2 \left[ x \gamma + y \eta - \mu \right], \\
    v_1 &= -a_1 \left[ x \gamma + y \eta - \mu \right], \\
    v_2 &= -a_2 \left[ x \gamma + y \eta - \mu \right].
\end{align*}
\]

Hence:

\[
\begin{align*}
    u &= a_1 \left[ x \gamma + y \eta - \mu \right], \\
    v &= a_2 \left[ x \gamma + y \eta - \mu \right],
\end{align*}
\]

the solutions of functions \( p \) and \( q \) (parallel to the two types of above-mentioned body waves), towards a function unifying their characteristics, may be obtained by stating:

\[
\begin{align*}
    \psi(u) &= A_0 (\phi(u)), \\
    \psi(v) &= B_0 (\phi(u)),
\end{align*}
\]

Remembering that:

\[
\begin{align*}
    a_1 &= (\alpha) \beta, \\
    a_2 &= (\alpha) \beta,
\end{align*}
\]

express the surface velocity of the two types of waves, and that:

\[
\begin{align*}
    \gamma_1 &= \tan \alpha_1, \\
    \gamma_2 &= \tan \alpha_2,
\end{align*}
\]

\( \alpha_1 \) and \( \alpha_2 \) being the respective angles of emergence, one obtains (\( \lambda \)) (page 236):

\[
\begin{align*}
    \frac{1}{a_1} &= \frac{1}{a_2} \left( \frac{1}{a_1} \frac{1}{a_2} \right).
\end{align*}
\]

Hence we shall have, before interference of the two types of waves:

\[
\begin{align*}
    s_1 &= \frac{1}{a_1} \left[ x \gamma + y \eta - \mu \right], \\
    s_2 &= \frac{1}{a_2} \left[ x \gamma + y \eta - \mu \right], \\
    v_1 &= \frac{1}{a_1} \left[ x \gamma + y \eta - \mu \right], \\
    v_2 &= \frac{1}{a_2} \left[ x \gamma + y \eta - \mu \right].
\end{align*}
\]
But on the surface \( s = 0 \):

\[
(\nu^3)_{1} = (\nu^3)_{2} = \nu^3,
\]

so that, stated \( u_0 = u_1 + u_2 \), \( u_0 = \omega_1 + \omega_2 \), for \( s = 0 \) we will have:

\[
\begin{align*}
\alpha_1 & = \frac{1}{2\nu_1 \nu_2} + \frac{\omega_1 (\nu - \omega_0)}{\nu_1 (\nu - \omega_0)} \Phi (\nu - \omega_0) \\
\alpha_2 & = \frac{1}{2\nu_1} - \frac{\nu_1 (\nu - \omega_0)}{\nu_1 (\nu - \omega_0)} \Phi (\nu - \omega_0),
\end{align*}
\]

which represents the result of the unifying of the pairs of body waves \( \varphi \) and \( \psi \) in the surface wave \( \varphi' x, \psi' y \), which is generated by their superposition.

From \([1]\) it follows that:

\[
\begin{align*}
\nu_1 & = \nu - \frac{\beta}{\gamma} \\
\alpha & = \nu_1, \quad \omega_1 = \nu_2 \\
\omega & = \nu_1, \quad \omega_2 = \nu_2.
\end{align*}
\]

It is to be recalled \([4]\) that:

\[
\begin{align*}
\tan \beta_1 & = \frac{\gamma - 1}{\gamma + 1} \\
\tan \beta_2 & = \frac{\gamma - 1}{\gamma + 1},
\end{align*}
\]

where \( \gamma \) stands for one of the two real roots of the Rayleigh equation greater than 1.

From the equation of condition \([7]\):

\[
[1 + (1 + 2 \nu) \nu \omega_0] \nu_0 (\nu - \omega_0) = 4 \nu_1 \nu_2 \nu \omega_0,
\]

it follows that it has to be:

\[
\nu_1 = \pm \sqrt{\frac{\gamma - 1}{\gamma + 1}} \quad \text{and} \quad \nu_2 = \frac{\gamma - 1}{\gamma + 1},
\]

that is, as has already been observed \([7]\) (page 236) only couplings of a direct longitudinal wave and its own reflected transverse wave, or
of a direct transverse wave and its relative reflected longitudinal wave satisfy the Rayleigh equation: this is required by the starting conditions of the Somigliana theory [see (1), page 227, formula (3)].

Hence, the (1) are to be written:

\[
\begin{align*}
a_0 &= \pm \frac{1}{2a} \left( \frac{\eta Z - 1}{\eta_0 (Z - Z_0)} \right) \Phi (x - x_0) \\
b_0 &= \frac{1}{2a} \frac{1}{\eta_0 (Z - Z_0)} \Phi (x - x_0)
\end{align*}
\]

[5]

Now, if the \( \phi (x - x_0) \) is a periodic function oscillating between \(-\xi\) and \(\xi\), while \(2\eta_0\) and \(2\eta_1\) are the given amplitudes for \(u_0\) and \(u_1\), respectively, to determine \(u_0\) and \(u_1\), we shall have the linear equations:

\[
\begin{align*}
\pm \frac{1}{2a} \left( \frac{\eta_0 Z - 1}{\eta_1 (Z - Z_0)} \right) &= \frac{2\eta_0}{\eta_1 (Z - Z_0)} \Phi (x - x_0) \\
\frac{1}{2a} \frac{1}{\eta_0 (Z - Z_0)} &= \frac{2a_1}{2a_0 (Z - Z_0)} \Phi (x - x_0)
\end{align*}
\]

[6]

which may serve the purpose, provided the determinant of the coefficients of \(2a_0\) and \(2a_1\) be other than 0.

When \( \phi (x - x_0) \) is not periodic, the relation \(\eta_0/\eta_1\) may be determined with formula (3) under due consideration of formulae (4).

2. - It is known that the value of \(\alpha\) for the surface layers of the Earth's crust, does not differ appreciably from 0.25.

R. Gutenberg(7) places it between 0.25 and 0.27 for depth up to 50 km, assuming for this depth a mean value of 0.26.

When \(x = 1\) the Rayleigh equation gives for real \(x\) (greater than 1) the two values:

\[
\lambda = \frac{n_0}{\eta_0} \left( \frac{1.0}{\lambda_0}, \frac{3.1517}{\lambda_0} \right).
\]
So in the case of an incident longitudinal wave, the determinant of the coefficients of $\ddot{x}$ and $\ddot{\alpha}$ is easily seen as being:

$$\frac{1}{x} - \left( \frac{1}{x} + \sqrt{x^2 - 1} \right)^2,$$

(and likewise for incident transverse wave, except substitution of the symbols).

It is quickly seen that for $x = \sqrt{2}$, system $[6]$ is not valid since the determinant equals zero, therefore $\omega_1 = 0$ and $\omega_2 = 0$. In other words when $x = \sqrt{2}$, the Somigliana waves are generated only in correspondence with:

$$\gamma = \frac{\omega^2}{\omega^2} = 3.1547.$$

Consequently, the values of $\gamma$, with which one can expect waves of Somigliana, may generally vary between 3.1547 when $a = \frac{1}{3}$ and $3.5754$ when $a = 0.2630$. The apparent velocity $v_3$ of these waves must therefore always be inferior to $2v_2$.

As for the effective angles, we may assume (based on values obtained by resolving Rayleigh equation for $a$ included between the limits 0.0 and 0.5) a mean angle of incidence on the order of 35° for incident longitudinal waves, and an angle of 20° in the case of incident transverse waves.

With regard to the $C_{ij}$ waves, therefore, the effective angles at the bottom of the various layers should not differ widely from the value of 35°. This calls for a gradually increasing distance from the source of the $C_{ij}, C_{ii}$ and $C_{ij}$, as indeed is the case. For Somigliana waves generated by incident longitudinal waves — except for particular cases — the starting distances would be decidedly shorter.

3. The $C_{ij}$: We shall not devote too much space to the waves of this type, as a great deal has already been published on the subject (2.5, 0.7).

Again we want to state that, when the conditions for their formation exist, these waves may reach remarkable extensions. It is possible to have them recorded, notwithstanding their long period,
by instruments of short period and small static magnification (smoked-paper recording).

The examples cited in a study of 1934 (8), where the SL are to be understood as C\textsubscript{ij} and the SM as C\textsubscript{ij}, have been almost all obtained by smoked recording instruments, with periods shorter than 30 seconds.

Those reproduced in two notes of 1948 (9) and 1949 (10) refer to C\textsubscript{ij} waves (with periods on the order of 30 seconds), recorded (Utica, New Brunswick, Mexico) by smoked recording instruments having periods of only 2 seconds. Hence, indications are for evident forced waves of remarkable amplitude, capable of impressing even seismographs with almost no sensitivity for them. In fact, the C\textsubscript{ij} waves constitute the largest waves of the whole seismogram, including the surface waves.

Further documentation is reported in Figures 2, 3, 4, 5, 6.

The apparent velocities of the C\textsubscript{ij} waves have been determined previously. Also the apparent velocities of the C\textsubscript{ij} waves have been the object of inquiry in the development of a doctoral thesis (Ranalli 1963).

The calculation of the apparent velocity of the C\textsubscript{ij} waves seemed to be somewhat more complex, also because the confocal angle to their generation, if beyond the 8,000 km, is generally associated with PS instead of with S waves (Figg. 4, 5).

The calculation of the velocities of seismic waves presumes the availability of a number of sufficient data obtained from one and the same earthquake. Initially we disposed of data gathered from various earthquakes, obtained by preceding researches and from a recent systematic research, carried out on a certain number of earthquakes properly to collect the greatest possible number of examples of C\textsubscript{ij} waves.

Even though keeping in mind the limitations due to the non-homogeneity of the data with regard to the origin of the earthquakes, we considered the possibility to utilize them for an attempt to assess the velocities of C\textsubscript{ij} waves. Specifically, in view of the wide field of epicentral distances covered by the above mentioned data and considering that for distances beyond the 8,000 km the C\textsubscript{ij} waves appear associated with the PS waves, the calculation was carried out by subdividing the collected data in two groups:

a) epicentral distances below 8,000 km;

b) epicentral distances beyond 8,000 km.
Fig. 2 - With governing conditions observed, the $C_{ij}$ waves are apt to reach remarkable amplitude, even in instruments with short period. In the above example (Trieste 18/VI/1933; $\delta - 5^\circ.3N$; $\beta - +5^\circ.5N$; $\gamma - 38^\circ.5N$; $\eta - 142^\circ.8E$; $H = 21^\circ.37m$; $a = 9400$ km) the $C_{ij}$ waves have been recorded by a seismograph with period of about 10 sec. Seismographs, using smoked-paper, with period on the order of 1 second, have supplied clear examples of $C_{ij}$ waves.
Fig. 3 When the conducive angles are reached in correspondence to the seismic stations, the Somigliana waves are recorded almost simultaneously with the incident longitudinal waves or transverse waves with which they are associated. The example shown above of Co1 and Co2 waves, obtained at Halifax is particularly significant.
At distances over 8000 km, the $C_n$ waves are generally associated with $P_S$ waves, as shown in the above example (Aquila 28/V.63: Ep. Krolski 291: 407 N, 33°40' E, $D = 8000$ km).
Fig. 6. Where the Earth's crust is of lower thickness, these P waves result of much shorter periods than those reached in continental areas. The example above has been obtained on the coasts of Greenland.
The equation obtained is:

\[ Y = \left( 0.39052 \pm 0.03823 \right) X + \left( 2.35148 \pm 1.35440 \right) \]

where \( Y \) and \( X \) represent the travel-times in minutes and the epicentral distances in degrees.

The relative velocity results of about 8.9 km/sec.

The data utilized:

<table>
<thead>
<tr>
<th>Station</th>
<th>Earthquake</th>
<th>( J )</th>
<th>Travel-time, sec.</th>
</tr>
</thead>
<tbody>
<tr>
<td>De Bilt</td>
<td>Karm 38 E, 1600 E</td>
<td>8700</td>
<td>19 15</td>
</tr>
<tr>
<td>Mooningits</td>
<td>Ambots 39 E, 1800 E</td>
<td>8675</td>
<td>19 15</td>
</tr>
<tr>
<td>Flensburg</td>
<td>Altenes 2, 1750 E, 2000 E</td>
<td>6970</td>
<td>10 35</td>
</tr>
<tr>
<td>Halifax</td>
<td>De Bilt 38 E, 1600 E</td>
<td>7417</td>
<td>20 15</td>
</tr>
<tr>
<td>Kew</td>
<td>Ambots 40 E, 1600 E</td>
<td>7743</td>
<td>20 22</td>
</tr>
<tr>
<td>Palisades</td>
<td>Ambots 39 E, 1800 E</td>
<td>7448</td>
<td>20 40</td>
</tr>
<tr>
<td>De Bilt</td>
<td>Central 40 E, 1600 E, 2000 E</td>
<td>7700</td>
<td>21</td>
</tr>
</tbody>
</table>

The equation obtained is:

\[ Y = (0.21997 \pm 0.01382) X + (5.839359 \pm 1.01480) \]

for a velocity of about 8.1 km/sec.

This value for \( v \) fully confirms what has been observed in \( n^2 \).

2. - The source of the \( C_n \) waves, as was exposed in the preceding notes (\( v \) does not appear indicated). The fact that in the Oceanic islands, they appear for periods much shorter than the mean one observed in continental stations — and, generally, only one of the three possible types (\( v \)) was already significant.
Fig. 7 - If the records are obtained on the bottom of the Ocean, where the Molierovic surface is at only a few km depth, the recording of the Cn, waves is almost missing. This is shown by the above recording obtained with a seismograph placed on the bottom of the Pacific Ocean, at the place of coordinates 38°09',2 N; 124°54',4 W, at a depth of 3860 m.
Fig. 8 - Some Venezuelan earthquake records, as Fig. 7, obtained in continental stations, with large Somigliana waves ($P$, $S$).
In view of these facts, it was to be expected that the recording of $C_1$ waves at the bottom of the ocean would have been hardly possible, if not in particular circumstances and with periods decidedly shorter. This is exactly what happened (see Fig. 7). The Venezuelan earthquake of July 29, 1967, from which is exceptionally large continental records of $C_1$ waves (Figs. 8, 8a), permitted confirmation of what had been anticipated. The amplitude of these waves, and their relative periods, appeared already reduced corresponding to the Greenland stations.

Here, the Earth's crust is of minor thickness and without apparent stratification. The waves are almost totally lacking on the bottom of the Pacific—near the epicenter of coordinates 38° 09' N, 124° 54' W—where the Ocean presents a depth of 3860 m ab. and the crustal thickness is estimated about 10 km. At almost the same epicentral distance (Toledo), instruments of the same type supplied $C_0$ waves of an uncommon amplitude (see Fig. 8). In Halifax, at the same distance of the cited suboceanic station from Venezuela, the $C_1$ and $C_2$ waves connected with the Alaskan earthquake of 24 VI. 1963 have been recorded with an exceptional amplitude (Fig. 9). Halifax is a continental
station, where the crust presents the layers called for to generate strong Somigliana waves.

5. - PL waves. The problem of long-period waves that followed closely after the first phase of a seismogram (or its reflections) was attacked for the first time in 1930 by Somville (12). He distinguishes them as PL waves. Again in 1931 this was confirmed by Somville and by Landsberg (13). In these notes, the Authors supply unequivocal examples of PL waves for epicentral distances below 2000 km, and they determine periods (> 10 seconds) and travel-time curves. In particular, Somville (12) points out that these waves, whose apparent velocity results equal to 8/10 of the velocity of the direct ones, are to be sharply distinguished from the first reflections of the longitudinal waves.

In 1960, Oliver and Major (14) supplied an interpretation as to the nature of the waves. According to the PL waves constitute a group of waves with normal dispersion rate and periods larger than 10 seconds, whose propagation proceeds via the "leaking mode", as would be derived from the correlation between dispersion of the PL and of the Rayleigh waves (11). The movement of the surface particle is always elliptical and progressive.

Furthermore, it is not to be excluded that such waves may be recorded at epicentral distances larger than 25 degrees.

Based on this interpretation, studies by Phinney (15) and by Gilbert and Lester (16) are carried out both theoretically and with the use of models. Subsequently Oliver (17), using studies of phase velocities of PL waves, recorded with long-period equipment in the United States, in connection with two earthquakes in Mexico of May 1962 (18, 1966 km), puts forth the possibility of utilizing these waves to obtain information on the structure of the Earth's crust, as it occurs with the Rayleigh and Love waves in a more obvious manner.

As has been already observed elsewhere (11), with regard to the PL waves, our explanation of their origin may well find its place in the modified theory of Somville. The effective angle for their generation is high (1,2 - 3). It follows that particular conditions are required for their generation. Therefore, because of the particular conditions of PL propa-

(*) This propagation is supposed to be associated with the crustal wave guide.
gitation, the near earthquakes are the most qualified for their formation. This, is undoubtedly the reason for the greater frequency of this type of waves with short epicentral distances.

The earthquake of Mistretta, occurred on October 31, 1907, supplied evidence of PL waves at Rome (d = 490 km, Fig. 10), at L’Aquila (d > 500 km, Fig. 11), and at Trieste (d > 870 km, Fig. 12). It is our opinion that these are Somigliana waves generated by the longitudinal waves. With regard to the hypoeutectic depth (about 10 km), as well as to the structure of the crust relative to the Tyrrhenian sea, the propagation of the P waves towards Rome should have proceeded as shown in Figure 10.

By analogy this fact must be valid for the recording at L’Aquila, but not at Trieste. In the latter the same type of wave is to be attributed to PP waves, incident at the bottom of the intermediate layer (or “granite” layer), under the same conductive angle. This type of wave is rarely observed in connection with distant earthquakes. Sometimes, however, even in case of teleseismic, the P waves appear followed by remarkable PL waves (see Fig. 13). We have observed that such records are associated, generally, with earthquakes beginning in the asthenosphere (i.e., where the velocity undergoes a flexion). This fact may cause the P (or PP) waves to present themselves at the bottom of the crust with very high angles of incidence (see Fig. 13) and thus permit the generation of Somigliana waves at the bottom of one of the layers of the crust by incidence of longitudinal waves under conducive angles. Considering the mechanism of propagation for seismic rays, influenced by the asthenosphere, it is possible that the PL waves, associated with earthquakes of distant origins, may be started—as Somigliana waves—by longitudinal waves that reach the seismic station slightly later than the normal P waves, which have crossed layers characterized by a higher propagation velocity.

With this mode of reason, we can also explain the long-period P waves which are sometimes recorded in connection with very near earthquakes. We refer to the example reproduced in Figure 14. The unusual length of the period of the P waves may be explained by the generation of Somigliana waves in the alluvial layer underneath the seismic station of Rome (Tiber valley), connected with the incidence under conducive angles of longitudinal waves at the bottom of the same layer.

The suggested theory justified also for the setting up of the long waves, which on great distances are associated with the SL,
Fig. 10. The hypocentral depth influences the starting conditions of the waves. The examples of PL waves reported above and in figures 11, 12, are due to the fact that hypocentral depth of the source of the earthquake was of about 70 km. Therefore it was possible to reach the conducive angles at Rome and at L’Aquila in correspondence to the P waves, and at Trieste in correspondence to the PP wave.
Fig. 11 - See Fig. 10
Fig. 12 - See Fig. 10
Fig. 13 - PL wave in distant earthquake.
(St. Louis, 11.VIII.1961, d = 8300 km).
Shallow earthquakes may set up P waves at short distances, as in the example recorded at Rome, in association with the earthquake occurring under the Laga mountains (3-XII-1967), at an epicentral distance of only 100 km. The maximum angle was probably made at the bottom of the thick alluvial layer of the Tiber valley.
Fig. 15 - The exposed theory justified also the formation of long waves, which at great distances are associated with $S$, $SS$, $SSS$, ..., waves, and for which there seems to be no other possible explanation. See also figures 16 and 17, in which the Co wave represents the largest phase of the whole seismogram.
waves, and for which there seems to be no other possible explanation. The examples shown in figures 15, 16, 17 proceed the largest phases of the whole seismogram for the $C_0$ waves. This would prove that $C_0$ waves have the same free oscillations of the Earth's crust layers, where they take origin.

REFERENCES

P. Calo, A. Romualdi


